ALGEBRAIC TOPOLOGY I WS23/24, HOMEWORK SHEET 5

DEADLINE: FRIDAY, NOVEMBER 17TH

Problem 1.

Part A. Consider the fibre sequence

$$S^0 \to S^1 \xrightarrow{p} S^1,$$

where $p : S^1 \to S^1$ denotes the 2-sheeted cover $p(z) = z^2$. Let $H_0(F_{(-)}, \mathbb{Z})$ denote the local coefficient system on S^1 induced by the fibration.

Show that $H_*(S^1, H_0(F_{(-)}, \mathbb{Z}))$ is not isomorphic to the homology $H_*(S^1, H_0(S^0, \mathbb{Z}))$ with the constant coefficient system $H_0(S^0, \mathbb{Z})$.

Part B. Let X be a connected CW-complex with basepoint $x \in X$ and universal cover $q : \widetilde{X} \to X$. We suppose that $\pi_1(X, x)$ acts on another CW-complex Y. This also induces an action on homology,

$$\alpha : \pi_1(X, x) \times H_*(Y, \mathbb{Z}) \to H_*(Y, \mathbb{Z}).$$

Now consider the fibre bundle

$$Y \to \widetilde{X} \times_{\pi_1(X,x)} Y \xrightarrow{q \times x} X,$$

with $\pi_1(X, x)$ acting on $\widetilde{X}$ through deck transformations and $\widetilde{X} \times_{\pi_1(X,x)} Y$ is defined as the quotient of $\widetilde{X} \times Y$ by the diagonal $\pi_1(X, x)$-action. (You do not have to show that this is a fibre bundle.)

As described in the lecture, the homologies of the fibres form a functor on the fundamental groupoid of X. In particular, the fundamental group $\pi_1(X, x)$ acts on the homology of the fibre at the point x, which canonically identifies with Y. Let $\beta : \pi_1(X, x) \times H_*(Y, \mathbb{Z}) \to H_*(Y, \mathbb{Z})$ denote this action.

Show that the two actions α and β on homology are the same.

Problem 2. Prove the following, making use of the Serre spectral sequence:

Theorem. (Leray-Hirsch). Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration, with B path-connected. Assume that there exists a set of classes $\{c_j\} \in H^*(E, \mathbb{Z})$, of which only finitely many lie in a given degree, such that their restrictions $\{i^*(c_j)\}$ form a $\mathbb{Z}$-basis for the cohomology $H^*(F, \mathbb{Z})$ of the fiber F. Then the set $\{c_j\}$ is a basis of $H^*(E, \mathbb{Z})$ as a module over the ring $H^*(B, \mathbb{Z})$.